RATIONAL EXPRESSIONS

Rational expressions can be described as a polynomial fraction, or as the ratio of two polynomials.

Examples:
\[
\frac{x}{x + 2} \quad \frac{x^2 - 2x + 4}{x^4 + 6x^2 + 8}
\]

Working with rational expressions is similar to working with fractions. Simplify the fraction as much as possible to work with simpler terms. In the case of rational expressions we simplify by factoring each polynomial and canceling terms.

Reducing Rational Expressions to the Lowest Term

To reduce a rational expression to its lowest term, factor each polynomial and cancel like terms that are both in the numerator and the denominator.

Examples: Simplify \( \frac{x - 2}{x^2 - 4} \) → Factor each polynomial, if possible.
\[
\frac{x - 2}{(x + 2)(x - 2)} \rightarrow \text{Cancel like terms.}
\]
\[
= \frac{1}{x + 2}
\]

Reduce \( \frac{x^2 - 9}{x^2 + 6x + 9} \) to the lowest terms
\[
\frac{x^2 - 9}{x^2 + 6x + 9} = \frac{(x + 3)(x - 3)}{(x + 3)(x + 3)}
\]
\[
= \frac{x - 3}{x + 3}
\]

Multiplication of Rational Expressions
Multiplying two rational expressions is done in the same way as fraction multiplication. Multiply the numerators together and the denominators together. However, it is easier if you factor each polynomial before multiplying them together. If you multiply two polynomials you’ll end up with a really big polynomial, therefore it is easier to first factor and then combine.

Example:

Multiply \( \frac{x+2}{x^2 - 9} \) and \( \frac{x-3}{x^2 - 4} \)

\[
\frac{x+2}{x^2 - 9} \cdot \frac{x-3}{x^2 - 4} = \frac{(x+2)(x-3)}{(x^2 - 9)(x^2 - 4)} \rightarrow \text{Factor each polynomial completely.}
\]

\[
= \frac{(x+2)(x-3)}{(x-3)(x+3)(x-2)(x+2)} = \frac{1}{(x+3)(x-2)}
\]

Division of Polynomials

As with multiplication, division of polynomials is similar to fractions. The same rules apply. Multiply the first rational expression by the reciprocal of the second expression. Remember to factor each polynomial before multiplying to simplify the polynomials.

Example:

\[
\frac{x^2 - 5x - 6}{x^2 - 2x - 3} \div \frac{2x^2 - 11x - 6}{x^2 - 9} \rightarrow \text{Take the reciprocal of the second fraction and multiply the expressions.}
\]

\[
= \frac{x^2 - 5x - 6}{x^2 - 2x - 3} \cdot \frac{x^2 - 9}{2x^2 - 11x - 6} \rightarrow \text{Factor each polynomial completely.}
\]

\[
= \frac{(x-6)(x+1)}{(x-3)(x+1)} \cdot \frac{(x+3)(x+3)}{(x-6)(2x+1)} \rightarrow \text{Cancel common factors.}
\]

\[
= \frac{x+3}{2x+1}
\]
Addition and Subtraction of Rational Expressions

To add or subtract rational expressions it is necessary to first find the LCD, or least common denominator, as with fractions. The LCD for a set of rational expressions is the smallest quantity divisible by each of the denominators.

Examples: Add: \( \frac{x-1}{x^2 - 4} + \frac{x+2}{x^2 - 5x + 6} \)

Step 1 – Factor both denominators.

\[ x^2 - 4 = (x - 2)(x + 2) \quad x^2 - 5x + 6 = (x - 2)(x - 3) \]

Step 2 – The LCD will consist of each different term obtained from the factors of both denominators.

The LCD is: \( (x - 2)(x + 2)(x - 3) \)

Step 3 – Change each rational expression to an equivalent expression having the LCD as its denominator. Multiply the missing term from the LCD to the numerator.

\[ \frac{(x-1)(x-3)}{(x-2)(x+2)(x-3)} \rightarrow \text{The LCD is} (x - 2)(x + 2)(x - 3); \]

\[ \text{the missing factor here is} \ (x - 3). \]

\[ \frac{(x+2)(x+2)}{(x-2)(x+2)(x-3)} \rightarrow \text{The LCD is} (x - 2)(x + 2)(x - 3); \]

\[ \text{the missing factor here is} \ (x + 2). \]

Step 4 – Now that we have a common denominator, an LCD, we can add the two rational expressions together.

\[
\frac{(x-1)(x-3)}{(x-2)(x+2)(x-3)} + \frac{(x+2)(x+2)}{(x-2)(x+2)(x-3)} = \frac{(x-1)(x-3)+(x+2)(x+2)}{(x-2)(x+2)(x-3)} \rightarrow \text{Eliminate parentheses by distribution.}
\]

\[
= \frac{x^2 - 3x - x + 3 + x^2 + 2x + 2x + 4}{(x-2)(x+2)(x-3)} \rightarrow \text{Add common terms}
\]

\[
= \frac{2x^2 + 7}{(x-2)(x+2)(x-3)} \rightarrow \text{Final answer.}
\]
Subtract: \[ \frac{2x - 1}{x^2 + 5x + 6} - \frac{7}{x^2 - x - 12} \]

**Step 1** – Factor both denominators.

\[ x^2 + 5x + 6 = (x + 3)(x + 2) \]
\[ x^2 - x - 12 = (x - 4)(x + 3) \]

**Step 2** – The LCD will consist of each different term obtained from the factors of both denominators.

The LCD is \( (x + 3)(x + 2)(x - 4) \)

**Step 3** – Change each rational expression to an equivalent expression having the LCD as its denominator. Multiply the missing term from the LCD to the numerator.

\[ \frac{2x - 1}{x^2 + 5x + 6} = \frac{2x - 1}{(x + 3)(x + 2)} \cdot \frac{(x - 4)}{(x - 4)} = \frac{(2x - 1)(x - 4)}{(x + 3)(x + 2)(x - 4)} \]
\[ \frac{7}{x^2 - x - 1} = \frac{7}{(x - 4)(x + 3)} \cdot \frac{(x + 2)}{(x + 2)} = \frac{7(x + 2)}{(x + 3)(x + 2)(x - 4)} \]

**Step 4** – Now that we have a common denominator, an LCD, we can add the two rational expressions together.

\[ \frac{(2x - 1)(x - 4)}{(x + 3)(x + 2)(x - 4)} - \frac{7(x + 2)}{(x + 3)(x + 2)(x - 4)} = \frac{(2x - 1)(x - 4) - 7(x + 2)}{(x + 3)(x + 2)(x - 4)} \]
\[ = \frac{2x^2 - 8x - x + 4 - 7x - 14}{(x + 3)(x + 2)(x - 4)} \]
\[ = \frac{2x^2 - 16x - 10}{(x + 3)(x + 2)(x - 4)} \]
\[ = \frac{2(x^2 - 8x - 5)}{(x + 3)(x + 2)(x - 4)} \]
RATIONAL EXPRESSIONS – EXERCISES

Reduce to lowest terms:

1. \( \frac{x - 2}{x^2 - 4} \)
2. \( \frac{5x + 25}{x^2 - 25} \)
3. \( \frac{x^2 - 2x + 1}{x - 1} \)

4. \( \frac{x - 3}{x^2 - 6x + 9} \)
5. \( \frac{x^2 - 4}{x^2 - 4x + 4} \)
6. \( \frac{2x^2 + 5x - 3}{x^2 - 9} \)

Perform the indicated operations:

7. \( \frac{x - 3}{x^2 - 4} \cdot \frac{x + 2}{x^2 - 6x + 9} \)
8. \( \frac{x + y}{x - 1} \cdot \frac{x^2 - 2x + 1}{x^2 - y^2} \)

9. \( \frac{3x^2 - 2x - 8}{2x^2 + 3x - 2} + \frac{x^2 - 4}{3x + 4} \)
10. \( \frac{x^2 + 7x + 12}{x - 5} \div \frac{x^2 + 9x + 18}{x^3 - 7x + 10} \)

11. \( \frac{x + 3}{2x - 1} + \frac{x - 1}{2x - 1} \)
12. \( \frac{1}{3x^2} + \frac{5}{2x^3} \)

13. \( \frac{1}{x - 3} + \frac{3}{x^3 - 27} \)
14. \( \frac{x}{x^2 - 6x + 9} + \frac{3}{2x^2 - 5x - 3} \)

15. \( \frac{2}{x^2 - x - 12} - \frac{4}{x^2 + 6x + 9} \)
16. \( \frac{x}{2x^2 - 3x - 20} - \frac{1}{2x^2 + 7x + 5} \)
RATIONAL EXPRESSIONS – ANSWERS TO EXERCISES

1. \[ \frac{x-2}{x^2-4} = \frac{x-2}{(x-2)(x+2)} = \frac{1}{x+2} \]

2. \[ \frac{5x+25}{x^2-25} = \frac{5(x+5)}{(x-5)(x+5)} = \frac{5}{x-5} \]

3. \[ \frac{x^2-2x+1}{x-1} = \frac{(x-1)(x-1)}{x-1} = x-1 \]

4. \[ \frac{x-3}{x^2-6x+9} = \frac{x-3}{(x-3)(x-3)} = \frac{1}{x-3} \]

5. \[ \frac{x^2-4}{x^2-4x+4} = \frac{(x-2)(x+2)}{(x-2)(x-2)} = \frac{x+2}{x-2} \]

6. \[ \frac{2x^2+5x-3}{x^2-9} = \frac{(x+3)(2x-1)}{(x-3)(x+3)} = \frac{2x-1}{x-3} \]

7. \[ \frac{x-3}{x^2-4} \cdot \frac{x+2}{x^2-6x+9} = \frac{(x-3)(x+2)}{(x-2)(x+2)(x-3)(x-3)} = \frac{1}{(x-2)(x-3)} \]

8. \[ \frac{x+y}{x-1} \cdot \frac{x^2-2x+1}{x^2-y^2} = \frac{(x+y)(x-1)(x-1)}{(x-1)(x+y)(x-y)} = \frac{x-1}{x-y} \]

9. \[ \frac{3x^2-2x-8}{2x^2+3x-2} + \frac{x^2-4}{3x+4} = \frac{3x^2-2x-8}{2x^2+3x-2} \cdot \frac{3x+4}{x^2-4} \]
\[ = \frac{(3x+4)(x-2)(3x+4)}{(2x-1)(x+2)(x-2)(x+2)} = \frac{(3x+4)^2}{(2x-1)(x+2)^2} \]

10. \[ \frac{x^2+7x+12}{x-5} \div \frac{x^2+9x+18}{x^2-7x+10} = \frac{x^2+7x+12}{x-5} \cdot \frac{x^2-7x+10}{x^2+9x+18} \]
\[ = \frac{(x+4)(x+3)(x-5)(x-2)}{(x-5)(x+6)(x+3)} = \frac{(x+4)(x-2)}{x+6} \]
11. \[ \frac{x+3}{2x-1} + \frac{x-1}{2x-1} = \frac{(x+3)+(x-1)}{2x-1} = \frac{2x}{2x-1} = \frac{2(x+1)}{2x-1} \]

12. \[ \frac{1}{3x^2} + \frac{5}{2x^3} = \frac{2x}{6x^3} + \frac{15}{6x^3} = \frac{2x+15}{6x^3} \]

13. \[ \frac{1}{x-3} + \frac{3}{x^3-27} = \frac{1}{x-3} + \frac{3}{(x-3)(x^2+3x+9)} \]

\[ = \frac{1(x^2+3x+9)}{(x-3)(x^2+3x+9)} + \frac{3}{(x-3)(x^2+3x+9)} = \frac{x^2+3x+12}{(x-3)(x^2+3x+9)} \]

14. \[ \frac{x}{x^2-6x+9} + \frac{3}{2x^2-5x-3} = \frac{x}{(x-3)(x-3)} + \frac{3}{(2x+1)(x-3)} \]

\[ = \frac{x(2x+1)+3(x-3)}{(x-3)(x-3)(2x+1)} = \frac{2x^2+4x-9}{(x-3)(x-3)(2x+1)} = \frac{2x^2+4x-9}{(x-3)^2(2x+1)} \]

15. \[ \frac{2}{x^2-x-12} - \frac{4}{x^2+6x+9} = \frac{2}{(x-4)(x+3)} - \frac{4}{(x+3)(x+3)} \]

\[ = \frac{2(x+3)-4(x-4)}{(x-4)(x+3)(x+3)} = \frac{-2x+22}{(x-4)(x+3)(x+3)} = \frac{-2(x-11)}{(x-4)(x+3)^2} \]

16. \[ \frac{x}{2x^2-3x-20} - \frac{1}{2x^2+7x+5} = \frac{x}{(2x+5)(x-4)} - \frac{1}{(2x+5)(x+1)} \]

\[ = \frac{x(x+1)-1(x-4)}{(2x+5)(x-4)(x+1)} = \frac{x^2+4}{(2x+5)(x-4)(x+1)} \]