POLYNOMIALS

Polynomials can be defined as the sum or difference of terms or expressions. Each term can be either a constant or variable, have one or more terms, and be composed of like terms or different terms.

The expressions that can exist in a polynomial are defined as:

**Integer**- a negative or positive whole number including zero. Example: -5, -4, -1, 0, 1, 5, 6, 7 etc...

**Constants** – A single number in the equation that does not contain any variable. *Example: 4, 6*

**Coefficient** – The numerical part of a monomial. *Example: 7 is the coefficient of 7x^3y*

**Degree** – The highest power to which a variable is raised.

*Examples:*

Identify each term in the following polynomial: \(2x^3 - 4\)

- **Constant:** - 4 → the single number in the equation
- **Coefficient:** 2 → the number in front of the variable \(x\)
- **Degree:** 3 → the highest power of the variable

Give the degree of the following polynomial: \(5x^6 - 3x^4 - 11x^2 + 8\)

**Degree:** 6 → It is a sixth degree polynomial because the highest exponent of \(x\) is 6.

There are also different types of polynomials:

**Monomial** – A constant, or the product of a constant, and one or more variables raised to an integer. *Example: -3x^2y^3z*

**Polynomial** – Any finite sum (or difference) of terms. *Example: 4x^3y^2 - 3z + 9x^2y - 2xz^3*

**Binomial** – A polynomial consisting of exactly two terms. *Example: 2x - 7*

**Trinomial** – A polynomial consisting of exactly three terms. *Example: x^3 - x + 4*

There are special binomial rules that can be followed that can make them easier to solve.

\[
(a + b)^2 = a^2 + 2ab + b^2 \quad \quad (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3
\]

\[
(a - b)^2 = a^2 - 2ab + b^2 \quad \quad (a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3
\]

\[
(a + b)(a - b) = a^2 - b^2
\]
Examples: \((x + 2)^2\) → Solve using the binomial properties

\[
(a + b)^2 = a^2 + 2ab + b^2
\]
\[
(x + 2)^2 = x^2 + 2(x)(2) + 2^2
\]
\[
(x + 2)^2 = x^2 + 4x + 4
\]

\((y - 3)^3\) → Solve using the binomial properties

\[
(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3
\]
\[
(y - 3)^3 = y^3 - 3(y)^2(3) + 3(y)(3)^2 - 3^3
\]
\[
(y - 3)^3 = y^3 - 9y^2 + 27y - 27
\]

Combining like Terms

Polynomials can be short expression or really long ones. Probably the most common thing you will be doing with polynomials is combining like terms in order to simplify long polynomial expressions. Combining “like terms” is the process by which we combine exact same terms containing the same variable with the same exponent together to shorten the expression.

Examples:

\[3x + 4x\] → Both terms contain the same variable. Therefore, they can be combined.
\[3x + 4x = 7x\]

\[2x^2 + 3x - 4 - x^2 + x + 9\] → Combine like terms together and then simplify.
\[2x^2 + 3x - 4 - x^2 + x + 9 = \]
\[(2x^2 - x^2) + (3x + x) + (-4 + 9)\]
\[x^2 + 4x + 5\]
Evaluating Polynomials

To evaluate polynomials substitute the variables for the given value and solve the equation.

Examples: Evaluate \(2x^3 - x^2 - 4x + 2\) at \(x = -3\).

\[
2(-3)^3 - (-3)^2 - 4(-3) + 2 \quad \rightarrow \quad \text{Plug in } -3 \text{ for } x; \text{ be careful with parentheses and negative signs.}
\]

\[
= 2(-27) - (9) + 12 + 2 \\
= -54 - 9 + 12 + 2 \\
= -63 + 14 \\
= -49
\]

If \(P(x) = 5x^2 - 4x + 7\), find \(P(2)\). \(\rightarrow\) This is the same thing as plugging a 2 for ‘x’

\[
P(2) = 5(2)^2 - 4(2) + 7 \\
= 5(4) - 8 + 7 \\
= 20 - 8 + 7 \\
= 19
\]

Addition and Subtraction of Polynomials

The addition and subtraction of polynomials consist of combining like terms by grouping together the same variable terms with the same degrees. In the case of subtraction, if the subtraction sign (or negative sign) is outside parentheses, the first thing to do is to distribute the negative sign to each of the terms inside the parentheses.

Examples:

Simplify: \((3x^3 + 3x^2 - 4x + 5) + (x^3 - 2x^2 + x - 4)\)

\[
(3x^3 + 3x^2 - 4x + 5) + (x^3 - 2x^2 + x - 4) \quad \rightarrow \quad \text{Combine same degree terms together.}
\]

\[
= 3x^3 + 3x^2 - 4x + 5 + x^3 - 2x^2 + x - 4
\]

\[
= 4x^3 + x^2 - 3x + 1
\]

Simplify: \((2x + 5y) + (3x - 2y)\)
2x + 5y + 3x − 2y → Combine x’s with x’s and y’s with y’s.

(2x + 3x) + (5y − 2y)

5x + 3y

Simplify: \((x^3 + 3x^2 + 5x - 4) - (3x^3 - 8x^2 - 5x + 6)\)

\((x^3 + 3x^2 + 5x - 4) - (3x^3 - 8x^2 - 5x + 6)\)

\(x^3 + 3x^2 + 5x - 4 - 3x^3 + 8x^2 + 5x - 6\)

\((x^3 - 3x^3) + (3x^2 + 8x^2) + (5x + 5x) + (-4 - 6)\)

\(= -2x^3 + 11x^2 + 10x - 10\)

**Multiplication of Polynomials**

To multiply polynomials the same rules apply as with exponents since we are dealing with variables with different exponents. Remember that when multiplying, the exponents add together.

**Example:** \((3x^3)(10x^2)\) → Multiply the constant terms together first and then the variables.

\(= (3 \cdot 10)(x^3 \cdot x^2)\) → Add exponents together.

\(= 30x^5\)

Next is the one-term polynomial times a multi-term polynomial. Distribute the one-term polynomial outside the parentheses to each term inside the polynomial.

**Example:** \(-3x(4x^2 - x + 10)\) → Distribute the \(-3x\) to all the terms inside the parentheses.

\(= -3x(4x^2) - 3x(-x) - 3x(10)\)

\(= -12x^3 + 3x^2 - 30x\)

Multiplication of two two-term polynomials is a little more complex. To make it easier we use the **FOIL method**: a method that simplifies the process by following a pattern. FOIL stands for First, Outer, Inner and Last. Start with the first terms in each parenthesis, followed by the outer terms, then the inner terms, and finally the last terms. This method makes it easier to remember which terms to multiply and reduces the chance of forgetting to multiply some terms.

**Example:**

Use the **FOIL** method to simplify \((2x + 5)(x + 3)\)

"first": \((2x)(x) = 2x^2\)

\((2x + 5)(x + 3)\)
In order to multiply one multi-term polynomial by another multi-term polynomial, break the smaller polynomial apart and multiply each individual term by the longer polynomial.

Example:

\[(x + 2)(x^3 + 3x^2 + 4x - 17)\] → Take the smaller polynomial and multiply each of its individual terms by the largest polynomial.

\[x^4 + 3x^3 + 4x^2 - 17x + 2x^3 + 6x^2 + 8x - 34\]
\[= x^4 + 5x^3 + 10x^2 - 9x - 34\]

**Division of Polynomials**

When dealing with polynomials divided by a single term the division can be treated as a simplification problem and the action is just to reduce its terms to the lowest possible.

Examples:

Simplify: \(\frac{2x + 4}{2}\) → In this case, there is a common factor in the numerator (top) and denominator (bottom), so it is easy to reduce this fraction.

There are two ways of proceeding. Split the division into two fractions, each with only one term on top, and then reduce:

\[\frac{2x + 4}{2} = \frac{2x}{2} + \frac{4}{2} = x + 2\]

Or factor out the common factor from the top and bottom and then cancel:

\[\frac{2x + 4}{2} = \frac{2(x + 2)}{x} = x + 2\]

Either way, the answer is the same: \(x + 2\)

Simplify: \(\frac{18x^4 + 36x^3}{9x}\)
Method 1:  \[
\frac{18x^4 + 36x^3}{9x} = \frac{18x^4}{9x} + \frac{36x^3}{9x} = 2x^3 + 4x^2
\]

Method 2: If you recognize a common factor that can be taken out of the parentheses and used to cancel the term in the denominator use it.

\[
\frac{9x(2x^3 + 4x^2)}{9x} = 2x^3 + 4x^2
\]

The answer is the same by any method: \(2x^3 + 4x^2\)

**Long Division**

If you come about a more complicated polynomial division you can use the long division method to do the operation. Long division of polynomial works just as a regular long division, with the exception that in this case variables are included.

**Example:** Divide \(x^2 - 6x - 12\) by \(x + 2\).

**Step 1 – Set up the division.**

\[
x + 2 \overline{x^2 - 6x - 12}
\]

**Step 2 – Let’s look at the first term inside, \(x^2\). On the outside we have an x, so to get to \(x^2\) we need to multiply \(x \cdot x\). Write this term on top.**

\[
x + 2 \overline{x^2 - 6x - 12}
\]

**Step 3 – Multiply the term on the top by the term outside and write it at the bottom, \(x(x + 2) = x^2 + 2x\). However, to be able to eliminate the terms we need to change the sign, \(-x^2 - 2x\).**

\[
x + 2 \overline{x^2 - 6x - 12}
\]

\[
x + 2 \overline{-x^2 - 2x}
\]

**Step 4 – Perform the required operations.**
Step 5 – Bring down the next term inside the division, in this case \(-12\).

\[
\begin{array}{c}
x + 2 | x^2 - 6x - 12 \\
-x^2 - 2x \\
\hline
-8x - 12
\end{array}
\]

Step 6 – Repeat step 2 through 4 to eliminate the remaining terms.

\[
\begin{array}{c}
x - 8 \\
\hline
x + 2 | x^2 - 6x - 12 \\
-x^2 - 2x \\
\hline
-8x - 12 \\
8x + 16 \\
\hline
4
\end{array}
\]

The solution to the long division is \(x - 8\).

Divide \(3x^3 - 5x^2 + 10x - 3\) by \(3x + 1\):

\[
\begin{array}{c}
x^2 - 2x + 4 \\
\hline
3x + 1 | 3x^3 - 5x^2 + 10x - 3 \\
-3x^3 - x^2 \\
\hline
-6x^2 + 10x \\
6x^2 + 2x \\
\hline
12x - 3 \\
-12x - 4 \\
\hline
-7
\end{array}
\]

\(\leftarrow x^2 (3x + 1) = 3x^3 + x^2\), to subtract, \(-\left(3x^3 + x^2\right) = -3x^3 - x^2\)

\(\leftarrow -2x (3x + 1) = -6x^2 - 2x\), to subtract, \(-\left(-6x^2 - 2x\right) = 6x^2 + 2x\)

\(\leftarrow 4(3x + 1) = 12x + 4\), to subtract, \(-\left(12x + 4\right) = -12x - 4\)

Since the division did not come out even, the answer is the polynomial on top of the division plus the remainder as a fraction with the divisor as the denominator: \(x^2 - 2x + 4 + \frac{-7}{3x + 1}\).

**POLYNOMIALS – EXERCISES**
Perform the indicated operations:

1. \((7x + 4) + (8x - 3)\)
2. \((x^2 + 6x + 3) + (4x^2 - 2x - 7)\)
3. \((x^4 + 1) + (-3x^3 + 3x^2 + 10)\)
4. \((2 - x) + (x^2 + 2) + (3x + 2)\)
5. \((x^3 + 2x^2 - 6x + 9) + (x^2 - 2x + 7)\)
6. \((x^2 - 1 + 0.08) - (-x^2 + 3x + 6)\)
7. \((8x^2 + 13x - 7) - (4x^2 + 7x - 1)\)
8. \((2x^2 - 5x + 4) - (4x^2 - 7x - 1)\)
9. \((x^2 + 6x + 2) + (2x^2 - 5x - 3) - (-3x^2 + 5x + 5)\)
10. \((x^2 - 2x + 1) - (3x + 2) - (2x^2 + 7)\)
11. \((x + 1)(4x + 3)\)
12. \((2x + 3)(x^2 + 3x + 1)\)
13. \((4x - 1)(3x^2 + 7x - 2)\)
14. \((2x - 3y)(x^2 + xy + 2y^2)\)
15. \((x + 6y)(2x^2 - xy - 2y^2)\)

Solve by long division:

16. \((4x^2 - 8x + 2) ÷ (2x)\)
17. \((8x^3 + 16x^2 + 20x + 4) ÷ (4x)\)
18. \((7x^2 - 2x + 8) ÷ (x + 1)\)
19. \((x^2 - 5xy - 6y^2) ÷ (x + y)\)
20. \((8x^2 - 2x - 15) ÷ (4x + 5)\)

POLYNOMIALS – ANSWER TO EXERCISES
1. \((7x + 4) + (8x - 3) = 15x + 1\)
2. \((x^2 + 6x + 3) + (4x^2 - 2x - 7) = 5x^2 + 4x - 4\)
3. \((x^4 + 1) + (-3x^3 + 3x^2 + 10) = x^4 - 3x^3 + 3x^2 + 11\)
4. \((2 - x) + (x^2 + 2) + (3x + 2) = x^2 + 2x + 6\)
5. \((x^3 + 2x^2 - 6x + 9) + (x^2 - 2x + 7) = x^3 + 3x^2 - 8x + 16\)
6. \((x^2 - 10x + 8) - (-x^2 + 3x + 6) = 2x^2 - 13x + 2\)
7. \((8x^2 + 1 - 3) - (4x^2 + 7x - 1) = 4x^2 + 6x - 6\)
8. \((2x^2 - 5x + 4) - (4x^2 - 7x - 1) = -2x^2 + 2x + 5\)
9. \((x^2 + 6x + 2) + (2x^2 - 5x - 3) - (-3x^2 + 5x + 5) = 6x^2 - 4x - 6\)
10. \((x^2 - 2x + 1) - (3x + 2) - (2x^2 + 7) = -x^2 - 5x - 8\)
11. \((x + 1)(4x + 3) = 4x^2 + 7x + 3\)
12. \((2x + 3)(x^2 + 3x + 1) = 2x^3 + 9x^2 + 11x + 3\)
13. \((4x - 1)(3x^2 + 7x - 2) = 12x^3 + 25x^2 - 15x + 2\)
14. \((2x - 3y)(x^2 + xy + 2y^2) = 2x^3 - x^2y + xy^2 - 6y^3\)
15. \((x + 6y)(2x^2 - xy - 2y^2) = 2x^3 + 11x^2y - 8xy^2 - 12y^3\)
16. \((4x^2 - 8x + 2) + (2x) = 2x - 4 + \frac{2}{2x}\)
17. \((8x^3 + 16x^2 + 20x + 4) ÷ (4x) = 2x^2 + 4x + 5 + \frac{4}{4x}\)
18. \((7x^2 - 2x + 8) ÷ (x + 1) = 7x - 9 + \frac{17}{x + 1}\)
19. \((x^2 - 5xy - 6y^2) ÷ (x + y) = x - 6y\)
20. \((8x^2 - 2x - 15) ÷ (4x + 5) = 2x - 3\)