Annuities

An annuity is a financial plan characterized by periodic payments/deposits. We can view an annuity as a savings plan in which the regular payments are contributions to the account, or we can also view an annuity as a payment plan in which regular payments are made from an account. Distinct formulas can be used to calculate the different types of annuities.

**Future Value of an Ordinary Annuity:**
If an amount \( R \) is deposited at the end of each period for \( n \) periods in an annuity that earns interest at a rate of \( i \) per period, then the future value of the annuity will be:

\[
S = R \cdot \left[ \frac{(1+i)^n - 1}{i} \right]
\]

- \( S \) = Future Value
- \( R \) = Payment
- \( i \) = Interest rate according to the period. Can be calculated by: \( i = \frac{x}{y} \)
- \( n \) = total time period, calculated by: \( n = t \cdot y \)

**Example:** $200 is deposited at the end of each quarter in an account that pays 4%, compounded quarterly. How much money will we have in the account in 2 years and 3 months?

**Solution:**

\[
S = 200 \cdot \left[ \frac{(1+0.04)^9 - 1}{0.01} \right] = 1873.71
\]

**Future Value of an Annuity Due**
If the payments/deposits of the annuity are given at the beginning of each time period, it is called an annuity due. The following formula calculates future value:

\[
S_{\text{due}} = R \cdot \left[ \frac{(1+i)^n - 1}{i} \right] (1+i)
\]

- \( S_{\text{due}} \) = Future Value
- \( R \) = Payment
- \( i \) = Interest rate according to the period. Can be calculated by: \( i = \frac{x}{y} \)
- \( n \) = total time period, calculated by: \( n = t \cdot y \)

**Example:** Find the future value of an investment if $150 is deposited at the beginning of each month for 9 years and the interest rate is 7.2%, compounded monthly.

**Solution:**

\[
R = $150, n = 9(12) = 108, i = 0.072/12 = 0.006 \quad S = 150 \cdot \left[ \frac{(1+0.006)^{108} - 1}{0.006} \right] (1+0.006) = 22,836.59
\]

**Present Value of an Ordinary Annuity**
To calculate the present value of an ordinary annuity the following formula is used:

\[
A_n = R \cdot \left[ \frac{1-(1+i)^{-n}}{i} \right]
\]

- \( A_n \) = Present Value
- \( R \) = Payment
- \( i \) = Interest rate according to the period. Can be calculated by: \( i = \frac{x}{y} \)
- \( n \) = total time period, calculated by: \( n = t \cdot y \)
Example: What is the present value of an annuity of $1,500 payable at the end of each 6-month period for 2 years if money is worth 8%, compounded semiannually?

Solution: \[ R = 1,500, i = 0.08 / 2 = 0.04, n = (2)(2) = 4 \]
\[
A_n = 1500 \left[ \frac{1 - (1 + 0.04)^{-4}}{0.04} \right] = 5,444.84
\]

**Present Value of an Annuity Due**

To calculate the present value of an annuity due the following formula is used:

\[
A_{(n,\text{due})} = \text{Present Value} = R \cdot \left[ \frac{1 - (1 + i)^{-n}}{i} \right] (1 + i)
\]

\( R = \) Payment

\( i = \) Interest rate according to the period. Can be calculated by \( i = \frac{x}{y} \)

\( n = \) total time period, calculated by: \( n = t \times y \)

Example: Suppose that a court settlement results in a $750,000 award. If this is invested at 9% compounded semiannually, how much will it provide at the beginning of each half-year for a period of 7 years?

Solution: \[
A_{(n,\text{due})} = 750,000, n = 2(7) = 14, i = 0.09 / 2 = 0.045
\]
\[
750,000 = R \left[ \frac{1 - (1 + 0.045)^{-14}}{0.045} \right] (1 + 0.045)
\]
\[
R = \frac{750,000}{10.682852} = 70,205.97
\]

**Deferred Annuity**

A deferred annuity is characterized by a payment which is made at some later date, rather than the beginning or end of the time period. The following formula calculates the present value of the deferred annuity:

\[
A_{(n,k)} = \text{Present Value} = R \cdot \left[ \frac{1 - (1 + i)^{-n}}{i} \right] (1 + i)^{-k}
\]

\( A_{(n,k)} = \) Present Value

\( R = \) Payment

\( i = \) Interest rate according to the period. Can be calculated by \( i = \frac{x}{y} \)

\( n = \) total time period, calculated by: \( n = t \times y \)

\( k = \) deferred periods

Example: A deferred annuity is purchased that will pay $10,000 per quarter for 15 years after being deferred for 5 years. If money is worth 6% compounded quarterly, what is the present value of this annuity?

Solution: \[
R = 10,000, n = 4(15) = 60, k = 4(5) = 20, i = 0.06 / 4 = 0.015
\]
\[
A_{(60,20)} = 10,000 \left[ \frac{1 - (1 + 0.015)^{-60}}{0.015} \right] (1.015)^{-20} = 292,386.85
\]