Finding the Equation of a Circle

By definition, a circle is the set of all points \( P(x, y) \) whose distance from a center \( C(h, k) \) is the distance \( r \). Thus \( P \) is considered a point on the circle if and only if the distance from \( P \) to \( C \) equals \( r \). The general equation of a circle is 

\[ r^2 = (x - h)^2 + (y - k)^2 \]

where \( r \) is the radius and the point \((h, k)\) is the center of the circle.

Example 1: Find the equation of a circle with radius=3 and a center \( (2, -5) \).

Step 1 Begin with the general equation of a circle:

\[ r^2 = (x - h)^2 + (y - k)^2 \]

Step 2 Plug in the values in place of their corresponding variables:

*From the above example, \( r = 3, h = 2 \) and \( k = -5 \).*

Therefore: 

\[(3)^2 = (x - (2))^2 + (y - (-5))^2\]

Step 3 Simplify the equation:

Simplify \((3)^2 = (x - (2))^2 + (y - (-5))^2 \) \( \ldots \) to get \( 9 = (x - 2)^2 + (y + 5)^2 \)

Example 2: Find the equation of a circle that has points \( P(1, 8) \) and \( Q(5, -6) \) as the endpoints of a diameter.

Step 1 Use the midpoint formula, 

\[ m = \frac{(x_1 + x_2)}{2}, \frac{(y_1 + y_2)}{2} \], to find the center of the circle:

a. Plug values into midpoint formula \( m = \frac{(1 + 5)}{2}, \frac{(8 + (-6))}{2} \)

b. Simplify \( m = \frac{6}{2}, \frac{2}{2} \) \( \ldots \) to get \( m = (3,1) \)

The center between \( P \) and \( Q \) is \( x = 3 \) and \( y = 1 \), which is also the center coordinates, \((h, k)\), of the circle.

Step 2 Use the general equation of a circle, one of the points given and the coordinates for the center to find the value of \( r \):

a. General equation of a circle is \( r = \sqrt{(x - h)^2 + (y - k)^2} \)

b. Plug in values \( P(1, 8) \) and \( C(3, 1) \) \( \Rightarrow r = \sqrt{(1-3)^2 + (8-1)^2} \)

c. Simplify \( r = \sqrt{(-2)^2 + (7)^2} \) \( \Rightarrow r = \sqrt{4 + 49} \) \( \Rightarrow r = \sqrt{53} \)

Thus, the radius of the circle is \( \sqrt{53} \).

Step 3 Plug in values into the general equation of a circle:

a. General equation of a circle is \( r^2 = (x - h)^2 + (y - k)^2 \)

b. Plug in values \( \Rightarrow (\sqrt{53})^2 = (x - 3)^2 + (y - 1)^2 \)

c. Simplify to get \( 53 = (x - 3)^2 + (y - 1)^2 \)
Example 3  Find the radius and center of a circle with equation \( x^2 + y^2 + 2x - 6y + 7 = 0 \).

Step 1  Group all x-terms and y-terms together, and move all constants to right-hand side of the equal sign:

a. The original equation is \( x^2 + y^2 + 2x - 6y + 7 = 0 \).
b. Group all x-terms and y-terms together \( \Rightarrow (x^2 + 2x) + (y^2 - 6y) + 7 = 0 \)
c. Move all constants to right-hand side of the equal sign \( \Rightarrow (x^2 + 2x) + (y^2 - 6y) = -7 \)

Step 2  Complete the square for each group by adding the square of half the coefficient of the x and y variable to each respective group and adding the same amount to the right-hand side of the equal sign:

a. The coefficient of \( x = 2 \) and of \( y = -6 \) \( \Rightarrow (x^2 + 2x + \left(\frac{2}{2}\right)^2) + (y^2 - 6y + \left(\frac{-6}{2}\right)^2) = -7 + \left(\frac{2}{2}\right)^2 + \left(\frac{-6}{2}\right)^2 \)

b. Simplify \( \Rightarrow (x^2 + 2x + 1) + (y^2 - 6y + 9) = -7 + 1 + 9 \Rightarrow (x^2 + 2x + 1) + (y^2 - 6y + 9) = 3 \)

c. Factor the resulting perfect square trinomials and write them as squares of a binomial \( \Rightarrow (x + 1)^2 + (y - 3)^2 = 3 \)

Step 3  Use the resulting equation to obtain the radius and coordinates for the center:

From the equation, \( (x + 1)^2 + (y - 3)^2 = 3 \), the center is \((-1, 3)\), and the radius is \(\sqrt{3} \).

Practice Exercises

Find the equation of a circle.

1. Center \((-1, -4)\); radius 8
2. Endpoints of a diameter are \(P (-1, 3)\) and \(Q (7, -5)\)

Given the equation of a circle, find the center and radius for each.

3. \( x^2 + y^2 - 2x - 2y = 2 \)
4. \( x^2 + y^2 + 6y + 2 = 0 \)

Answers:
1. \( (x + 1)^2 + (y + 4)^2 = 64 \)  2. \( (x - 3)^2 + (y + 1)^2 = 32 \)  3. Center \((1, 1)\); radius 2  4. Center \((0, -3)\); radius \(\sqrt{7} \)