De Moivré's Powers/Roots of Complex Numbers

De Moivré's Theorem may be used to find the n^{th} power and the n^{th} root of a complex number that is in the form $a+bi$.

The first step is to convert the complex number $a+bi$ into the trigonometric form $r(\cos\theta + i\sin\theta)$, where r is the modulus of $a+bi$ and

$$r = \sqrt{a^2 + b^2}$$

In addition, θ is the argument or the angle in standard position whose terminal side contains the point (a,b). We usually use the smallest nonnegative value for θ that satisfies

$$a = r\cos\theta \qquad\qquad a = r\sin\theta$$

$$\theta = \cos^{-1}\frac{a}{r} \qquad\qquad \theta = \sin^{-1}\frac{a}{r}$$

Example: Write the trigonometric form of $3-2i$.
Solution: From the complex number, it can be seen that $a = 3$ and $b = -2$, therefore,

$$r = \sqrt{3^2 + (-2)^2} = \sqrt{13}$$

and

$$\theta = \cos^{-1}\frac{3}{\sqrt{13}} = 33.7^o$$

Since the terminal side of θ must contain the point (3,-2), it must be in the fourth quadrant. Choose

$$\theta = 360^o - 33.7^o = 326.3^o$$

Hence,

$$3-2i = r(\cos\theta + i\sin\theta) = \sqrt{13}\left(\cos 326.3^o + i\sin 326.3^o\right)$$

Powers of Complex Numbers

De Moivré's Theorem states that if $z = r(\cos\theta + i\sin\theta)$ is a complex number and n is any positive integer, then

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Example: Find the 10^{th} power of $3-2i$ using De Moiré's theorem and write the answer in the form $a+bi$.
Solution: The 10^{th} power of $3-2i$ is

$$(3-2i)^{10}$$

From the previous example, the trigonometric form of $3-2i$ is

$$\sqrt{13}(\cos 326.3^o + i\sin 326.3^o)$$

therefore,

$$(3-2i)^{10} = \left[\sqrt{13}(\cos 326.3^o + i\sin 326.3^o)\right]^{10}$$

Applying De Moiré's Theorem where $z = 3-2i$,

$$(3-2i)^{10} = \sqrt{13}^{10}\left[(\cos(10\cdot 326.3^o) + i\sin(10\cdot 326.3^o))\right]$$

$$= 371{,}293\left[(\cos(3263^o) + i\sin(3263^o))\right]$$

Since the smallest angle was use, were

$$3263^o = 9\cdot 360 + 23$$

then,

$$\cos 3263^o = \cos 23^o = 0.9205$$

and

$$\sin 3263^o = \sin 23^o = 0.3907$$

Therefore,

$$(3-2i)^{10} = 371{,}293[0.9205 + i(0.3907)] = 341{,}775 + 145{,}064i$$

Roots of Complex Numbers

Theorem: If $z = r(\cos\theta + i\sin\theta)$ then it will have *n* distinct n^{th} roots, expressed as

$$r^{1/n}\left[\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right]$$

For $k = 0,1,2,\ldots,n-1$

The *n* distinct roots will have the same modulus, $r^{1/n}$, but will have different arguments determined by

$$\alpha = \frac{\theta+2k\pi}{n}$$

for radians and

$$\alpha = \frac{\theta+2k360^o}{n}$$

for degrees, where $k=0,1,2,\ldots,n-1$.

Example: Find the four fourth roots of $-8+8i\sqrt{3}$.

Solution: The modulus is

$$\sqrt{(-8)^2+\left(8\sqrt{3}\right)^2} = 16$$

and since the terminal side of θ contains the point $\left(-8,8\sqrt{3}\right)$, it must be in the second quadrant where

$$\cos\theta = \frac{-8}{16} = -\frac{1}{2} \text{ and } \theta = \cos^{-1}\left(\frac{-8}{16}\right) = 120^o$$

The four roots will then be evaluated by

$$16^{1/4}\left[\cos\left(\frac{120^o+k360^o}{4}\right)+i\sin\left(\frac{120^o+k360^o}{4}\right)\right]$$

Where $k=0,1,2,\ldots,n-1$ and since $n-1=3$, then $k=0,1,2,$and 3 and the 4 roots are

$$2\left[\cos 30^o + i\sin 30^o\right] = 2\left[\frac{\sqrt{3}}{2}+\frac{1}{2}i\right] = \sqrt{3}+i$$

$$2\left[\cos 120^o + i\sin 120^o\right] = 2\left[\frac{-1}{2}+i\frac{\sqrt{3}}{2}\right] = -1+i\sqrt{3}$$

$$2\left[\cos 210^o + i\sin 210^o\right] = 2\left[-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right] = -\sqrt{3}-i$$

$$2\left[\cos 300^o + i\sin 300^o\right] = 2\left[\frac{1}{2}-i\frac{\sqrt{3}}{2}\right] = 1-i\sqrt{3}$$